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Markov's Inequality and Zeros of Orthogonal Polynomials on Fractal Sets

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Zeros of orthogonal polynomials defined with respect to general measures are studied. It is shown that a certain estimate for the minimal distance between zeros holds if and only if the support F of the measure satisfies a homogeneity condition and Markov's inequality holds on F. © 1994 Academic Press, Inc.

INTRODUCTION

Let F be a compact subset of the m-dimensional Euclidean space $\mathbb{R}^m$. We say that F *preserves Markov's inequality* if there exist constants M and α such that

$$\| \, |\nabla P| \, \|_\infty \leqslant M n^\alpha \, \|P\|_\infty \tag{1}$$

for all polynomials P in m variables of degree $\leqslant n$, $n = 1, 2, 3, ...$, where ∇ denotes the gradient and the norm is the maximum norm on F. For the background to this concept, and its applications in approximation theory and in the theory of function spaces, see [6] and the references given there. Examples of sets preserving Markov's inequality are discussed in Section 2 below.

In [4] we pointed out a connection between sets preserving Markov's inequality and a property of orthogonal polynomials defined with respect to a measure μ with support F. Here we pursue this matter further by showing that, in the one-dimensional case and in the presence of a certain homogeneity condition, a set F preserves Markov's inequality if and only if the distances between consecutive zeros of the associated orthogonal polynomials satisfy a certain estimate. The results are given in Section 4 (Theorems 1, 2, and 3).

As a preparation for the proof of these results, we investigate in Section 1 an L^p-version of Markov's inequality, and give in Section 3 a related division inequality for polynomials.

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1. Markov's Inequality in $L^p(\mu)$

Throughout this section, F denotes a compact subset of $\mathbb{R}^m$. In the sequel, we will assume that to F is associated a probability measure μ with support F satisfying, for some constants $c_0 > 0$ and $s > 0$,

$$\mu(B(x, r)) \geqslant c_0 r^s, \qquad x \in F, \quad 0 < r \leqslant 1, \tag{2}$$

where $B(x, r)$ denotes the closed ball with center x and radius r; such a measure with $s = m$ exists on any compact set F, cf. [7]. Denote by $\|\cdot\|_p$ the norm in the Lebesgue space $L^p(\mu)$.

Definition 1. Let $1 \leqslant p \leqslant \infty$. A set F *preserves Markov's inequality in* $L^p(\mu)$ if there exist constants $M > 0$ and $\alpha > 0$ such that

$$\| \, |\nabla P| \, \|_p \leqslant M n^\alpha \|P\|_p \tag{3}$$

for all polynomials P in m variables of degree $\leqslant n$, $n = 1, 2, 3, \ldots$.

Clearly, F preserves Markov's inequality (as defined in the previous section) if and only if F preserves Markov's inequality in $L^\infty(\mu)$. If (3) is satisfied, we say that F preserves Markov's inequality in $L^p(\mu)$ with constants M and α and write $F \in \mathcal{M}^p(M, \alpha)$.

The condition imposed on F by Definition 1 is independent of p, as Proposition 1 below shows. The reason is that one can switch between norms as explained by the following lemma.

Lemma 1. *Let* $1 \leqslant p < \infty$.

(a) *If* $F \in \mathcal{M}^\infty(M, \alpha)$, *then it holds that for all polynomials P of degree* $\leqslant n$, $n = 1, 2, 3, \ldots$,

$$\|P\|_\infty \leqslant A_1 n^{\alpha s/p} \|P\|_p$$

with a constant $A_1 = A_1(M, m, p, c_0, s)$.

(b) *If* $F \in \mathcal{M}^p(N, \beta)$, *then it holds that for all polynomials P of degree* $\leqslant n$, $n = 1, 2, 3, \ldots$,

$$\|P\|_\infty \leqslant A_2 n^{\beta s/p} \|P\|_p,$$

with a constant $A_2 = A_2(N, m, p, c_0, s)$.

Proof. We first prove (a). Take $x_0 \in F$ with $|P(x_0)| = \|P\|_\infty$, and let v be a unit vector. Denote by D_v the directional derivative, and let

$x \in B(x_0, 1/(Mn^\alpha))$. Then a repeated use of (1) gives, using multi-index notation,

$$\begin{aligned}|D_v P(x)| &= \left| \sum_{|j| \leqslant n-1} \frac{D^j(D_v P)(x_0)}{j!} (x-x_0)^j \right| \\ &\leqslant \sum_{|j| \leqslant n-1} \frac{(Mn^\alpha)^{|j|+1} \|P\|_\infty}{j!\,(Mn^\alpha)^{|j|}} = Mn^\alpha\, |P(x_0)| \sum_{|j| \leqslant n-1} \frac{1}{j!} \\ &\leqslant Mn^\alpha e^m\, |P(x_0)|.\end{aligned}$$

Thus, by the mean value theorem, $|P(x)-P(x_0)| \leqslant Me^m n^\alpha\, |x-x_0|\, |P(x_0)|$, so if $x \in B(x_0, \delta)$ where $\delta = (2Me^m n^\alpha)^{-1}$, we have $|P(x)-P(x_0)| \leqslant |P(x_0)|/2$ and thus $|P(x)| \geqslant |P(x_0)|/2$. This gives $\|P\|_{p,\mu} \geqslant (\int_{B(x_0,\delta)} |P(x)|^p\, d\mu(x))^{1/p} \geqslant |P(x_0)|\, (\mu(B(x_0, \delta)))^{1/p}/2 \geqslant |P(x_0)|\, c_0^{1/p}\, \delta^{s/p}/2$, or, in view of our choice of x_0, $\|P\|_\infty \leqslant 2c_0^{-1/p}(2Me^m)^{s/p}\, n^{\alpha s/p}\, \|P\|_p$.

To prove (b), take again $x_0 \in F$ such that $|P(x_0)| = \|P\|_\infty$, and put $r = 1/(Nn^\beta)$. Then

$$\begin{aligned}\mu(B(x_0, r))\, |P(x_0)| &= \int_{|x-x_0| \leqslant r} |P(x_0)|\, d\mu(x) \\ &= \int_{|x-x_0| \leqslant r} \left| \sum_{|j| \leqslant n} \frac{(x_0-x)^j}{j!} D^j P(x) \right| d\mu(x) \\ &\leqslant \sum_{|j| \leqslant n} \frac{r^{|j|}}{j!} \int_{|x-x_0| \leqslant r} |D^j P(x)|\, d\mu(x) \\ &\leqslant \sum_{|j| \leqslant n} \frac{r^{|j|}}{j!} \left(\int_{|x-x_0| \leqslant r} |D^j P(x)|^p\, d\mu(x) \right)^{1/p} \\ &\quad \times \mu(B(x_0, r))^{1-1/p}.\end{aligned}$$

Using (3) repeatedly we get

$$\begin{aligned}\mu(B(x_0, r))^{1/p}\, |P(x_0)| &\leqslant \sum_{|j| \leqslant n} \frac{r^{|j|}}{j!} (Nn^\beta)^{|j|}\, \|P\|_p \\ &= \sum_{|j| \leqslant n} \frac{1}{j!} \|P\|_p \leqslant e^m\, \|P\|_p\end{aligned}$$

and thus by (2)

$$\|P\|_\infty \leqslant c_0^{-1/p} (Nn^\beta)^{s/p} e^m \|P\|_p.$$

Remark. To show that $|D_v P(x)| \leqslant Mn^\alpha e^m\, |P(x_0)|$ in the proof of statement (a) above, one does not need that $F \in \mathcal{M}^\infty(M, \alpha)$, but only the weaker condition that $\|D^j(D_v P)\|_\infty \leqslant (Mn^\alpha)^{|j|+1}\, \|P\|_\infty$, $|j| \leqslant n-1$. This means that the inequality $\|P\|_\infty \leqslant A_1 n^{\alpha s/p}\, \|P\|_p$ holds under this weaker condition.

We also note that in an analogous way as we obtained the estimate for $|D_v P(x)|$, one can get the following result (cf. [4, p. 310]). Let P be a polynomial of degree $\leqslant n$, and denote by $d(x, F)$ the distance from x to F. Assume that $\|D^j P\|_\infty \leqslant (Mn^\alpha)^{|j|} \|P\|_\infty$, $|j| \leqslant n$. Then $|P(x)| \leqslant e^{bMm} \|P\|_\infty$ for all x with $d(x, F) \leqslant b/n^\alpha$.

PROPOSITION 1. *Let $1 \leqslant p < \infty$. Then F preserves Markov's inequality if and only if F preserves Markov's inequality in $L^p(\mu)$. More precisely we have*

(a) *If $F \in \mathscr{M}^\infty(M, \alpha)$, then $F \in \mathscr{M}^p(N, \beta)$ with $\beta = \alpha(1 + s/p)$ and $N = N(M, m, p, c_0, s)$.*

(b) *If $F \in \mathscr{M}^p(N, \beta)$ then $F \in \mathscr{M}^\infty(M, \alpha)$ with $\alpha = \beta(1 + s/p)$ and $M = M(N, m, p, c_0, s)$.*

Proof. The result follows immediately from the previous lemma. For example, (b) follows from $\|D_v P\|_\infty \leqslant A_2(n-1)^{\beta s/p} \|D_v P\|_p \leqslant A_2 n^{\beta s/p} N n^\beta \|P\|_p \leqslant A_2 N n^{\beta(1+s/p)} \|P\|_\infty$.

2. Examples in the One-Dimensional Case

In this section we write, for short, $F \in \mathscr{M}$ if F preserves Markov's inequality.

In general, it is a non-trivial problem to decide if a set is in $\mathscr{M}$ or not. In several variables, large classes of sets in $\mathscr{M}$ are determined in [5]. In this paper we are mostly interested in the one-dimensional case, whence we discuss it in some detail.

Recall first that $\|P'\|_{\infty, F} \leqslant n^2/r \, \|P\|_{\infty, F}$ if F is an interval of length $2r$ (the norm is the maximum norm on F); this is the classical Markov inequality, and it shows that a finite union of closed intervals is in $\mathscr{M}$. We remark that Tjebychev polynomials are optimal for this inequality, and that one can easily use them to show that if $F \in \mathscr{M}(M, \alpha)$, then $\alpha \geqslant 2$. In [4] we showed that certain Julia sets (these are sets of Cantorian type which in general have Hausdorff dimension less than 1) are in $\mathscr{M}$, using the theory of orthogonal polynomials on Julia sets. It has also been shown, by A. Volberg, with a different method, that the Cantor ternary set is in $\mathscr{M}$; for the proof of this result, see [1]. It is of interest for this paper to recall that the Cantor ternary set has Hausdorff dimension $s = \log 2/\log 3$, and that a uniform measure supported on it satisfying (2) with this value on s is often used. Since these last two examples are quite difficult to treat, it is of some interest to point out how to obtain more elementary examples. We show that certain "thick" Cantor sets, with positive Lebesgue measure, are in $\mathscr{M}$; similar examples, using different methods, follow by results of J. Siciak, cf. [1].

We will use the Remez inequality. A somewhat weak form of it states that if P is a polynomial of degree $\leqslant n$ and $|P| \leqslant 1$ on a set E, where $E \subset I$, $I = [-1, 1]$, and $m(I \backslash E) \leqslant 2s$, $0 < s \leqslant 1/2$, then $|P| \leqslant e^{5n\sqrt{2s}}$ on I, see, e.g., [2]. If I is instead an arbitrary closed interval, the statement holds if one replaces the condition $m(I \backslash E) \leqslant 2s$ by $m(I \backslash E)/m(I) \leqslant s$; $0 < s \leqslant 1/2$; this follows by means of an affine change of variables.

Consider now the Cantor set $F = \bigcap_{v=1}^{\infty} F_v$ obtained as follows. Let $a > 0$, put $F_0 = [0, 1]$, and let F_v for $v \geqslant 1$ consist of 2^v intervals of equal length obtained by taking away from each interval in F_{v-1} an open interval of length $3^{-(a+1)v}$. Then $m(F_v) = 1 - \sum_{i=1}^{v} 2^{i-1}/3^{(a+1)i} = (3^{a+1} - 3 + (2/3^{a+1})^v)/(3^{a+1} - 2)$ so, denoting by I_v any one of the intervals in F_v,

$$c_1 2^{-v} \leqslant m(I_v), \tag{4}$$

where $c_1 = (3^{a+1} - 3)/(3^{a+1} - 2)$; note also that $m(F) = c_1$. Furthermore, $m(I_v \backslash F) = \sum_{i=v+1}^{\infty} 2^{i-v-1}/3^{(a+1)i} = 1/(3^{(a+1)v}(3^{a+1} - 2))$ so

$$\frac{m(I_v \backslash F)}{m(I_v)} = \frac{1}{3^{(a+1)v}} \cdot \frac{2^v}{3^{a+1} - 3 + (2/3^{a+1})^v} \leqslant \frac{1}{3^{a+1} - 3} \left(\frac{2}{3^{a+1}}\right)^v. \tag{5}$$

We now show that F preserves Markov's inequality. Assume that $\|P\|_{\infty, F} = 1$ and that the degree of P is $\leqslant n$. Let $\beta > 0$ and choose v so that $m(I_{v+1}) < 1/n^\beta \leqslant m(I_v)$. Then, by (4), $c_1 2^{-v-1} < 1/n^\beta$ and thus, by (5), defining γ by $3^{a+1} = 2^\gamma$,

$$\frac{m(I_v \backslash F)}{m(I_v)} \leqslant \frac{1}{(3^{a+1} - 3)\, 2^{(\gamma-1)v}} \leqslant \frac{1}{(3^{a+1} - 3)(2^{-1}c_1)^{\gamma-1}} \cdot \frac{1}{n^{\beta(\gamma-1)}} = \frac{c_2}{n^2},$$

where the equality was obtained by choosing $\beta = 2/(\gamma - 1)$ and the constant c_2 depends on a. Consequently, if $n \geqslant n_0$, where $n_0 = n_0(a)$ is so big that $c_2/n_0^2 \leqslant 1/2$, by Remez' inequality we have for $x \in I_v$ that $|P(x)| \leqslant e^{5n\sqrt{2c_2/n^2}} = c_3$. Since $m(I_v) \geqslant 1/n^\beta$, Markov's inequality gives $|P'(x)| \leqslant 2n^2 \|P\|_{\infty, I_v}/m(I_v) \leqslant 2n^{2+\beta}c_3 = 2c_3 n^{2+\beta} \|P\|_{\infty, F}$ for $x \in I_v$ and thus for $x \in F$. If $n < n_0$ we use this inequality with $n = n_0$ and get $|P'(x)| \leqslant 2c_3 n_0^{2+\beta} n^{2+\beta} \|P\|_{\infty, F}$, $x \in F$. Thus $F \in \mathcal{M}^\infty(M, \alpha)$ with $\alpha = 2 + 2/(\gamma - 1)$ and $M = 2c_3 n_0^{2+\beta}$. It is of some interest to recall that $\gamma = (a+1)\ln 3/\ln 2$ so $\gamma > \ln 3/\ln 2$, which gives an estimate for α independent of a; on the other hand M tends to infinity if a tends to zero.

3. A Division Inequality for Polynomials

For a more elaborate version of the following inequality, in the case when $F \subset \mathbb{R}^m$ and μ is the Lebesgue measure, see [3].

PROPOSITION 2. *Let $1 \leqslant p \leqslant \infty$, let $F \subset \mathbb{R}$ be a set preserving Markov's inequality in $L^p(\mu)$ with constants M and α, and let P be a polynomial of degree $n \geqslant 1$ of the form $P(x) = (x - x_0) P_1(x)$ where $x_0 \in \mathbb{R}$. Then it holds that*

$$\|P_1\|_p \leqslant 6Mn^\alpha \|P\|_p.$$

Proof. We take $\delta > 0$, and introduce the notation

$$_1\|f\| = \left(\int_{|x - x_0| \leqslant \delta} |f|^p \, d\mu\right)^{1/p}$$

and

$$_2\|f\| = \left(\int_{|x - x_0| > \delta} |f|^p \, d\mu\right)^{1/p}.$$

Then

$$\|P_1\|_p \leqslant {_1\|P_1\|} + {_2\|P_1\|}. \tag{6}$$

If $|x - x_0| > \delta$, then $|P_1(x)| = |P(x)|/|x - x_0| \leqslant |P(x)|/\delta$ so

$$_2\|P_1\| \leqslant \frac{1}{\delta} \|P\|_p. \tag{7}$$

Next, differentiating $P(x) = (x - x_0) P_1(x)$ one obtains $P'(x) = P_1(x) + (x - x_0) P_1'(x)$. Thus

$$\begin{aligned} _1\|P_1\| &= {_1\|P' - (x - x_0) P_1'\|} \leqslant {_1\|P'\|} + {_1\|(x - x_0) P_1'\|} \\ &\leqslant {_1\|P'\|} + \delta \, {_1\|P_1'\|} \leqslant \|P'\|_p + \delta \, \|P_1'\|_p \\ &\leqslant Mn^\alpha \|P\|_p + \delta M(n-1)^\alpha \|P_1\|_p. \end{aligned}$$

Together with (6) and (7) this gives

$$\|P_1\|_p \leqslant (Mn^\alpha + 1/\delta) \|P\|_p + \delta Mn^\alpha \|P_1\|_p.$$

Now we choose δ as $1/(2Mn^\alpha)$ which gives

$$\|P_1\|_p \leqslant 3Mn^\alpha \|P\|_p + \tfrac{1}{2} \|P_1\|_p, \qquad \text{so} \quad \|P_1\|_p \leqslant 6Mn^\alpha \|P\|_p.$$

Remark. The constant in the division inequality can be somewhat sharpened. For example, if $p = \infty$ one obtains by means of replacing (6) by $\|P_1\|_\infty \leqslant \max({_1\|P_1\|}, {_2\|P_1\|})$ the inequality $\|P_1\|_\infty \leqslant 2Mn^\alpha \|P\|_\infty$. We also remark that at least for $p = \infty$ the proposition admits a converse. Assume that $\|P_1\|_\infty \leqslant Ln^\alpha \|P\|_\infty$ for all P as in the proposition, let Q be a polynomial of degree n, and take $x_0 \in F$. Then, with $P_1(x) = (Q(x) - Q(x_0))/(x - x_0)$ we have $|Q'(x_0)| = P_1(x_0) \leqslant Ln^\alpha \|Q - Q(x_0)\|_\infty \leqslant 2Ln^\alpha \|Q\|_\infty$.

4. Zeros of Orthogonal Polynomials

In this section, F denotes a compact subset of $\mathbb{R}$. As before, we assume that a measure μ with support F satisfying (2) is given. Denote by P_n, $n = 0, 1, 2, \ldots$, orthogonal polynomials associated to μ with the degree of $P_n \neq n$. Normalization is of minor importance here, but we let them have leading coefficient 1; then the minimum of $\|P\|_2 = (\int |P|^2 \, d\mu)^{1/2}$ over all polynomials P of degree n with leading coefficient 1 is attained for $P = P_n$. Recall that the zeros of P_n are simple and real and situated in the smallest interval containing F. The theorems below explain a relation between Markov's inequality in the form we study it here and the distance between consecutive zeros of P_n.

THEOREM 1. *Assume that F preserves Markov's inequality in $L^2(\mu)$ with constants M and α, and let $n \geqslant 2$. Then there exists a constant $M_1 > 0$, depending on M only, such that if x_i and x_{i-1} are consecutive zeros of P_n, then*

$$|x_i - x_{i-1}| \geqslant \frac{M_1}{n^{2\alpha}}.$$

Proof. We assume that $[-1, 1]$ is the smallest interval containing F (the general case may be reduced to this by means of an affine change of coordinates). Then $P_n(x) = \prod_{i=1}^n (x - x_i)$, where the zeros x_i of P_n satisfy $|x_i| \leqslant 1$. We prove the theorem by means of a variation, showing that, for some M_1, if $|x_i - x_{i-1}| < M_1/n^{2\alpha}$, then there is a polynomial Q with leading coefficient 1, $Q \neq P_n$, with $\|Q\|_2 < \|P_n\|_2$, which is a contradiction.

Let $Q(x) = P_n(x)(x-a)(x-b)/((x - x_{i-1})(x - x_i))$, where $a < x_{i-1} < x_i < b$. Then $\int_a^b Q(x)^2 \, d\mu(x) \leqslant (b-a)^4 \int_{-1}^1 P_n(x)^2/((x - x_{i-1})^2 (x - x_i)^2) \, d\mu(x)$. Using the division inequality in Proposition 2 twice, first with $P_1(x) = P_n(x)/((x - x_{i-1})(x - x_i))$ and $x_0 = x_i$ and then with $P_1(x) = P_n(x)/(x - x_{i-1})$ and $x_0 = x_{i-1}$, we get, putting $L = 6M$, $\int_a^b Q(x)^2 \, d\mu(x) \leqslant (b-a)^4 L^2(n-1)^{2\alpha} L^2 n^{2\alpha} \|P_n\|_2^2 \leqslant L^4 (b-a)^4 n^{4\alpha} \|P_n\|_2^2$. Put now $x_i - x_{i-1} = 2\varepsilon$ and take $a = x_{i-1} - \varepsilon$ and $b = x_i + \varepsilon$. Then $(x-a)(x-b) = (x - x_{i-1} + \varepsilon)(x - x_i - \varepsilon) = (x - x_{i-1})(x - x_i) + \varepsilon(x - x_i - x + x_{i-1}) - \varepsilon^2 = (x - x_{i-1})(x - x_i) - 3\varepsilon^2$, and thus $Q(x) = P_n(x)(1 - r(x))$, where $r(x) = 3\varepsilon^2/((x - x_{i-1})(x - x_i))$. Denoting $[-1, 1] \setminus [a, b]$ by E, we have for $x \in E$

that $3\varepsilon^2 < (x - x_i)(x - x_{i-1}) < 4$ so $3\varepsilon^2/4 < r(x) < 1$ or $0 < 1 - r(x) < 1 - 3\varepsilon^2/4$ and thus, since $\varepsilon \leqslant 1$, $(1 - r(x))^2 < 1 - 3\varepsilon^2/2 + 9\varepsilon^4/16 \leqslant 1 - 15\varepsilon^2/16$. Consequently,

$$\int_E Q(x)^2 \, d\mu(x) = \int_E P_n(x)^2 \, (1 - r(x))^2 \, d\mu(x)$$

$$\leqslant \left(1 - \frac{15}{16} \varepsilon^2\right) \int_{-1}^{1} P_n(x)^2 \, d\mu(x).$$

Together with the estimate for $\int_a^b Q^2 \, d\mu$ above this gives, since $b - a = 4\varepsilon$,

$$\|Q\|_2^2 \leqslant \left((4\varepsilon L)^4 \, n^{4\alpha} + 1 - \frac{15}{16} \varepsilon^2\right) \|P_n\|_2^2 \, d\mu.$$

If $(4\varepsilon L)^4 \, n^{4\alpha} + 1 - (15/16) \, \varepsilon^2 < 1$, i.e., if $\varepsilon < \sqrt{15}/(4^3 L^2 n^{2\alpha})$, we have $\|Q\|_2^2 < \|P_n\|_2^2$ which is impossible. Thus we must have $|x_i - x_{i-1}| = 2\varepsilon \geqslant 2\sqrt{15}/(4^3 6^2 M^2) \cdot (1/n^{2\alpha})$, and the proof is finished.

Remark. If $F = [-1, 1]$, then we have, in view of the usual Markov's inequality and part (a) of Proposition 1, that $F \in \mathcal{M}^2(M, 2(1 + s/2)) = \mathcal{M}^2(M, 2 + s)$ so the theorem gives $|x_i - x_{i-1}| \geqslant M_1/n^{4+2s}$. If furthermore μ is the Lebesgue measure on $[-1, 1]$, then it is known that $F \in \mathcal{M}^2(M, 2)$ so the theorem gives the estimate $|x_i - x_{i-1}| \geqslant M_1/n^4$. This is poor compared to the exact information available in this case, but it should be noted that known, sharp, results concerning the distance between zeros deal with measures which are special compared to the ones considered here.

To prove a converse of Theorem 1, we need to assume a certain homogeneity condition. Let $\gamma \geqslant 1$. We say that F *satisfies the condition* (H_γ) if there exists a d with $0 < d \leqslant 1$ such that for any $x_0 \in F$ and any r, $0 < r \leqslant 1$, there is a point $x \in F$ satisfying

$$dr^\gamma \leqslant |x - x_0| \leqslant r.$$

In the proof of the next theorem the following lemma will be needed, cf. [4, Theorem 1].

LEMMA 2. *Assume that there are constants A and $\tau > 0$ such that $\|P_k'\|_\infty \leqslant Ak^\tau \, \|P_k\|_2$, $k = 1, 2, \ldots$. Then $F \in \mathcal{M}^\infty(A, \tau + 1/2)$.*

Proof. Let $\hat{P}_k$ denote the normalized polynomial $\hat{P}_k = P_k/\|P_k\|_2$. Then, by our assumption, $\|\hat{P}_k'\|_\infty \leqslant Ak^\tau$.

Let P be an nth degree polynomial. It may be written, with sums from $k=0$ to $k=n$, $P=\sum a_k \hat{P}_k$, where $(\sum |a_k|^2)^{1/2}=\|P\|_2$. Thus, for $x \in F$ we have

$$|P'(x)|=\left|\sum a_k \hat{P}'_k(x)\right| \leqslant \left(\sum |a_k|^2\right)^{1/2} \left(\sum |\hat{P}'_k(x)|^2\right)^{1/2}$$

$$\leqslant \|P\|_2 \left(\sum (Ak^\tau)^2\right)^{1/2} \leqslant An^{\tau+1/2} \|P\|_\infty,$$

which gives the lemma.

We also note that if I is an interval of length l, a repeated use of Markov's inequality $\|P'\|_{\infty, I} \leqslant 2n^2/l \, \|P\|_{\infty, I}$ gives, for an nth degree polynomial P,

$$\|P^{(j)}\|_{\infty, I} \leqslant (2n^2/l)^{|j|} \, \|P\|_{\infty, I}, \qquad |j| \leqslant n. \tag{8}$$

THEOREM 2. *Let $\gamma \geqslant 1$, assume that F satisfies (H_γ), and that for consecutive zeros x_{i-1} and x_i of P_n holds, for some $M, \alpha > 0$, and for $n \geqslant 2$,*

$$|x_i - x_{i-1}| \geqslant \frac{M}{n^\alpha}. \tag{9}$$

Then F preserves Markov's inequality.

Proof. We assume in the proof that the polynomials P_n are normalized so that $\|P_n\|_\infty = 1$. Take $n \geqslant 2$, and let $E=\{x; |P_n(x)| \leqslant 1\}$; then $F \subset E$. The set E consists of at most n disjoint, closed intervals, since P_n has n real zeros, and we denote by I_i the interval which contains x_i (possibly some I_i's are identical).

Assume that $I_i \cap F$ is not empty, $l(I_i) < dM^\gamma/n^{(\alpha+2)\gamma}$, and take $z \in I_i \cap F$. By the condition (H_γ) (we assume $M \leqslant 1$), there is a point $y \in F$ with $dM^\gamma/n^{(\alpha+2)\gamma} \leqslant |z-y| \leqslant M/n^{\alpha+2}$. Since y cannot be in I_i, and since $|y-x_i| \leqslant |y-z| + |z-x_i| \leqslant M/n^{\alpha+2} + dM^\gamma/n^{(\alpha+2)\gamma} \leqslant M/(2n^\alpha)$, it has to be in I_{i-1} or I_{i+1}, say in I_{i-1}. Then we have $l(I_{i-1}) \geqslant |x_{i-1}-y| \geqslant |x_{i-1}-x_i| - |x_i - y| \geqslant M/n^\alpha - M/(2n^\alpha) = M/(2n^\alpha)$. Thus, by (8), we have $\|P_n^{(j)}\|_{\infty, I_{i-1}} \leqslant (4n^{\alpha+2}/M)^{|j|} \, \|P_n\|_{\infty, I_{i-1}}$ for $|j| \leqslant n$, so by the remark given after the proof of Lemma 1 we have $|P_n(x)| \leqslant e^4 \, \|P_n\|_{\infty, I_{i-1}}$ if $d(x, I_{i-1}) \leqslant M/n^{\alpha+2}$. Since $y \in I_{i-1}$, and $|y-z| \leqslant M/n^{\alpha+2}$, it follows that $|P_n(x)| \leqslant e^4$ for $x \in [x_{i-1}, z]$, and thus for $x \in [x_{i-1}, z] \cup I_i = [x_{i-1}, x_i] \cup I_i$. In case $l(I_i) \geqslant dM^\gamma/n^{(\alpha+2)\gamma}$ we use that $|P_n| \leqslant 1$ on I_i, and thus we see that in any case there is an interval J_i containing I_i such that $l(J_i) \geqslant dM^\gamma/n^{(\alpha+2)\gamma}$ and $|P_n| \leqslant e^4$ on J_i. This holds also for $n=1$ since the diameter of F is $\geqslant d$ due to our assumption, and thus $\geqslant dM$ assuming $M \leqslant 1$.

Thus, by (8) we have

$$\|P_n^{(j)}\|_{\infty, I_i} \leqslant \|P_n^{(j)}\|_{\infty, J_i} \leqslant (2n^{(\alpha+2)\gamma+2}/dM^{\gamma})^{|j|} \|P_n\|_{\infty, J_i}, \qquad |j| \leqslant n,$$

and, consequently, since $\|P_n\|_{\infty, J_i} \leqslant e^4 = e^4 \|P_n\|_\infty \leqslant e^{4|j|} \|P_n\|_\infty$ for $|j| \geqslant 1$ and $F \subset E$, $\|P_n^{(j)}\|_\infty \leqslant ((2e^4/dM^{\gamma}) n^{(\alpha+2)\gamma+2})^{|j|} \|P_n\|_\infty$. By Lemma 1, and the remark given after its proof, we get

$$\|P_n\|_\infty \leqslant A_3(d, M, \gamma, c_0, s)\, n^{((\alpha+2)\gamma+2)s/2} \|P_n\|_2,$$

so $\|P_n'\|_\infty \leqslant A_4(d, M, \gamma, c_0, s)\, n^{((\alpha+2)\gamma+2)(1+s/2)} \|P_n\|_2$ and it follows from Lemma 2 that $f \in \mathscr{M}(A_4, (2+(\alpha+2)\gamma)(1+s/2)+1/2)$.

THEOREM 3. *Assume that F preserves Markov's inequality with constants M and α. Then satisfies (H_γ) for $\gamma > 2\alpha$.*

Proof. We may assume that F is contained in an interval of length 1. Recall from Section 2 that α satisfies $\alpha \geqslant 2$, so $\gamma > 2\alpha$ implies $\gamma > 1$. Take $\gamma > 1$, and assume that F does not satisfy (H_γ). Then one can find arbitrary small numbers r and points $x_0 \in F$, depending on r, such that $[x_0 - r, x_0 + r] \backslash (x_0 - r^\gamma, x_0 + r^\gamma)$ contains no points from F. (Otherwise there would exist an $r_0 \leqslant 1$ such that for every $r \leqslant r_0$ and $x_0 \in F$, there is a point $x \in F$ with $r^\gamma \leqslant |x - x_0| \leqslant r$. But then, if $1 \geqslant r \geqslant r_0$ and $x_0 \in F$, there is an $x \in F$ with $r \geqslant r_0 \geqslant |x - x_0| \geqslant r_0^\gamma \geqslant r_0^\gamma r^\gamma$, which means that F satifies (H_γ) with $d = r_0^\gamma$.) Take β with $1 < \beta < \gamma$. Corresponding to numbers x_0 and r as above, we construct polynomials Q of degree $n = n(r)$, with $n(r) \to \infty$ as $r \to 0$, such that (the norm is the maximum norm on F)

$$\|Q'\|_\infty \geqslant c n^{\gamma/(2\beta)} \|Q\|_\infty. \tag{10}$$

This means that we have $\alpha \geqslant \gamma/(2\beta)$ or $\gamma \leqslant 2\alpha\beta$, and since β can be taken arbitrary close to 1, we must have $\gamma \leqslant 2\alpha$. Thus F satifies (H_γ) for $\gamma > 2\alpha$.

Assume first that $x_0 = 0$. Define P by $P(x) = x(1-x^2)^k$. Then $P'(x) = (1 - x^2 - 2kx^2)(1-x^2)^{k-1}$, so $P'(0) = 1$ and $\pm 1/\sqrt{2k+1}$ are the zeros of P' in $(-1, 1)$. Take now k as the integer part of $r^{-2\beta}/2$; then $1/\sqrt{2k+2} < r^\beta \leqslant 1/\sqrt{2k}$, and it is not hard to see that, if r is small enough, the points $\pm 1/\sqrt{2k+1}$ lie in $[-r, r] \backslash (-r^\gamma, r^\gamma)$. Thus, since $F \subset [-1, 1]$, the maximum of $|P(x)|$ on F is less than or equal to the bigger of the numbers $P(r)$ and $P(r^\gamma)$.

Clearly, $P(r^\gamma) = r^\gamma (1 - r^{2\gamma})^k \leqslant r^\gamma$. Since $k > r^{-2\beta}/2 - 1$ we have, assuming $r \leqslant 1/2$ and noting that $(1-s)^{1/s} \leqslant e^{-1}$ if $0 < s < 1$, $P(r) = r(1-r^2)^k \leqslant (4/3)\, r(1-r^2)^{r^{-2\beta}/2} = (4/3)\, r(1-r^2)^{r^{-2} r^{2(1-\beta)}/2} \leqslant (4/3)\, r e^{-r^{2(1-\beta)}/2} \leqslant r^\gamma$, where the last inequality holds if r is small enough, since $\beta > 1$.

Thus if r is small enough we have, denoting by n the degree of P so $n = 2k+1$, $P'(0) = 1 = r^{-\gamma}r^{\gamma} \geqslant r^{-\gamma}\|P\|_\infty = (r^{-\beta})^{\gamma/\beta}\|P\|_\infty \geqslant (\sqrt{2k})^{\gamma/\beta}\|P\|_\infty \geqslant (1/2)\, n^{\gamma/(2\beta)}\|P\|_\infty$. Take now $Q(x) = P(x - x_0)$; then (10) is satisfied for small r, and the theorem is proved.

Note that the theorems above give the following characterization.

COROLLARY 1. *A set F preserves Markov's inequality if and only if there are constants $\gamma \geqslant 1$, $L > 0$, and $\beta > 0$, such that F satifies (H_γ) and such that if x_i and x_{i-1} are consecutive zeros of P_n, then it holds for $n \geqslant 2$ that $|x_i - x_{i-1}| \geqslant L/n^{\beta}$.*

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